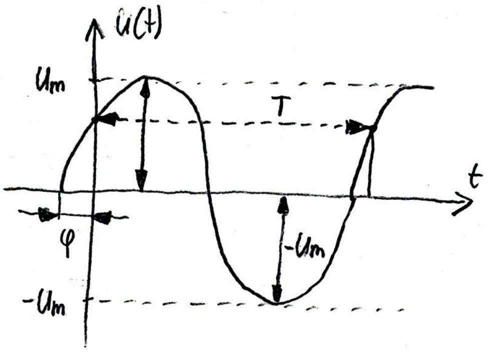
ELECTRICAL CIRCUITS 2 - CLASS NO. 1 (04.03.2025)

A SINE WAVE

$u(t) = U_m \cdot \sin(\omega t + \varphi)$



$u(t)$ - the instantaneous value of the signal

U_m - the maximum value of the signal
the peak value of the signal (amplitude)
the letter symbol for the peak value of an alternating source voltage is U_m (m for maximum)

T - period [s]

f - frequency [Hz] - the inverse of T, $f = \frac{1}{T}$

ω - angular velocity [rad/s] - radian per second

$\omega = 2\pi f = \frac{2\pi}{T}$

φ - phase angle or phase shift

U_{rms} - rms value (root-mean square), the effective value

$U_{rms} = \sqrt{\frac{1}{T}\int_0^T u^2(t)\,dt}$ $\qquad U_{rms} = \frac{U_m}{\sqrt{2}}$ the rms value of a sine wave of voltage is $1/\sqrt{2}$ of the peak value

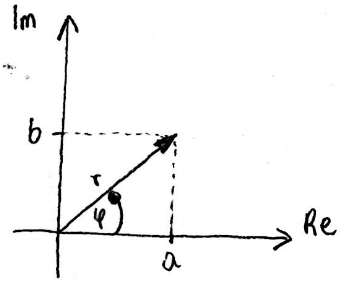
COMPLEX NUMBERS

* complex number can be represented in the form:
  - algebraic form (rectangular form): $\underline{z} = a + jb$
  - trigonometric form (polar form): $\underline{z} = r(\cos\varphi + j\sin\varphi)$
  - exponential form: $\underline{z} = r \cdot e^{j\varphi}$

* algebraic (rectangular) form of complex number

$\underline{z} = a + jb$

- complex numbers have a real part and an imaginary part
- a - real part ($Re(\underline{z})$) of complex number, $Re[z]$
- b - imaginary part of complex number, $Im[z]$
- j - imaginary unit (number) $j^2 = -1$, $j = \sqrt{-1}$ - square root of minus one
- graphical representation of complex number



- we use horizontal axis for the real part and vertical axis for the imaginary part
- $r = |z| = \sqrt{a^2 + b^2}$ - the absolute value of complex number (modulus)
- φ - argument of complex number $\varphi = \text{arc tg}\,\frac{b}{a}$

* <u>trigonometric (polar) form</u> of complex number

$$\underline{z} = r \cdot (\cos\varphi + j\sin\varphi)$$

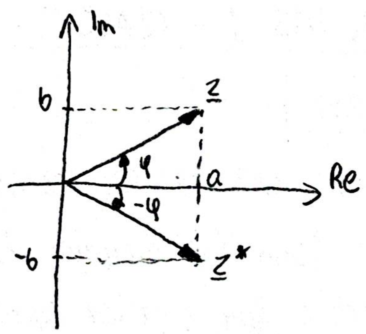
$r = \sqrt{a^2+b^2}$ $\quad \varphi = \operatorname{arc\,tg} \frac{b}{a}$ $\qquad \underline{z} = a + jb = r\left(\underbrace{\frac{a}{r}}_{\cos\varphi} + j\underbrace{\frac{b}{r}}_{\sin\varphi}\right) = r(\cos\varphi + j\sin\varphi)$

* <u>exponential form</u> of complex number

$\underline{z} = re^{j\varphi}$ $\qquad e^{j\varphi} = \cos\varphi + j\sin\varphi$

* <u>complex conjugate</u> of z

$$\left.\begin{aligned} \underline{z} &= a + jb \\ \underline{z} &= r(\cos\varphi + j\sin\varphi) \\ \underline{z} &= re^{j\varphi} \end{aligned}\right\} \Rightarrow \begin{aligned} \underline{z}^* &= a - jb \\ \underline{z}^* &= r(\cos\varphi - j\sin\varphi) \\ \underline{z}^* &= re^{-j\varphi} \end{aligned}$$



BASIC OPERATIONS ON COMPLEX NUMBERS

$\underline{z}_1 = a_1 + jb_1 \qquad \underline{z}_2 = a_2 + jb_2$

* <u>sum of complex numbers</u>

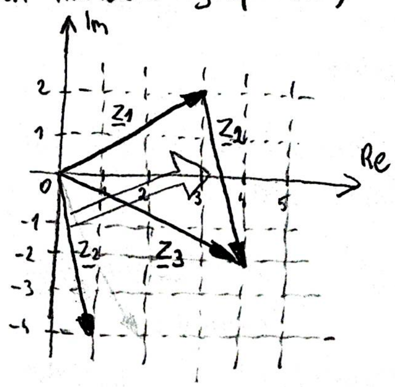
$$\underline{z} = \underline{z}_1 + \underline{z}_2 = a_1 + jb_1 + a_2 + jb_2 = \underbrace{a_1 + a_2}_{Re} + j\underbrace{(b_1 + b_2)}_{Im}$$

we can add complex numbers graphically

$\underline{z}_1 = 3 + j2$

$\underline{z}_2 = 1 - j4$

$\underline{z}_1 + \underline{z}_2 = 4 - j2$



o we move vector $\underline{z}_2$ to the end of vector $\underline{z}_1$

* <u>difference of complex numbers</u>

$$\underline{z} = \underline{z}_1 - \underline{z}_2 = \underbrace{a_1 - a_2}_{Re} + j\underbrace{(b_1 - b_2)}_{Im}$$

* <u>product of complex number</u>

$$\underline{z} = \underline{z}_1 * \underline{z}_2 = (a_1 + jb_1) * (a_2 + jb_2) = \underbrace{a_1 \cdot a_2 - b_1 \cdot b_2}_{Re} + j\underbrace{(a_1 b_2 + b_1 a_2)}_{Im}$$

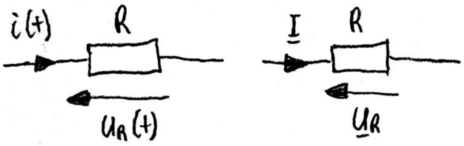
* <u>division of complex number</u>

$$\underline{Z} = \frac{\underline{Z}_1}{\underline{Z}_2} = \frac{a_1 + jb_1}{a_2 + jb_2} = \frac{(a_1+jb_1)\cdot(a_2-jb_2)}{(a_2+jb_2)\cdot(a_2-jb_2)} = \underbrace{\frac{a_1\cdot a_2 + b_1\cdot b_2}{a_2^2+b_2^2}}_{Re} + j\underbrace{\frac{b_1\cdot a_2 - a_1\cdot b_2}{a_2^2+b_2^2}}_{Im}$$

- we multiply numerator and denominator by the conjugate of the denominator

RESISTOR

* Resistor, Resistance [Ω]



$u_R(t) = R\cdot i(t)$

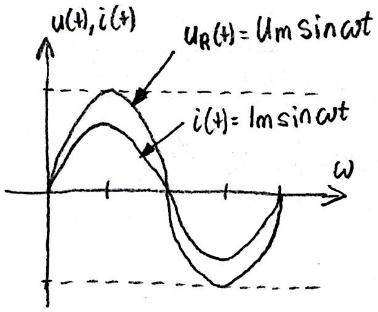
$\underline{U}_R = R\cdot \underline{I}$

R - resistance [Ω]

G - conductance [S]

Z - impedance

Y - admittance



phasor diagram

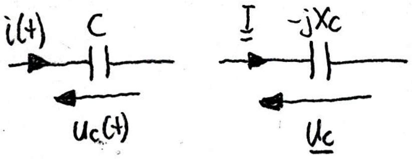
* the voltage across a resistance is <u>in phase</u> with the current

$\psi_u = \psi_i$

$G = \frac{1}{R}$ $\quad \underline{Z} = R \quad \underline{Y} = \frac{1}{\underline{Z}} = \frac{1}{R} = G$

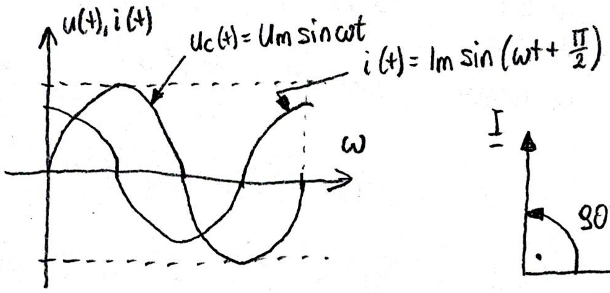
CAPACITOR

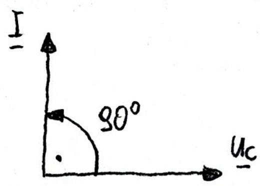
* Capacitor, Capacitance [F]



X_C - capacitive reactance [Ω]

B_C - capacitive susceptance [S]





phasor diagram

$X_C = \frac{1}{\omega C}$ $\quad \omega = 2\pi f$

$\underline{Z} = -jX_C \quad \underline{Y} = \frac{1}{-jX_C} = jB_C$

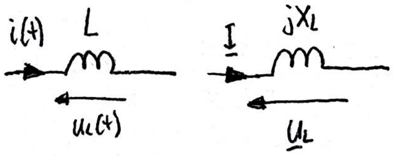
$u_C(t) = \frac{1}{C}\int i(t)dt \quad \underline{U}_C = -jX_C\cdot\underline{I}$

* the voltage across a capacitor <u>lags</u> the current by 90° ($\frac{\pi}{2}$)

$\psi_i = \psi_u + 90°$

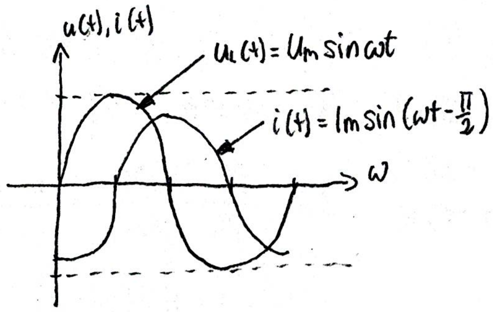
INDUCTOR

* Inductor, Inductance [H]



X_L - inductive reactance [Ω]

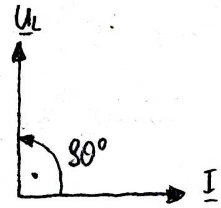
B_L - inductive susceptance [S]



$X_L = \omega L \qquad \omega = 2\pi f$

$\underline{Z} = jX_L \qquad \underline{Y} = \frac{1}{\underline{Z}} = \frac{1}{jX_L} = -jB_L$

$u_L(t) = L\frac{di(t)}{dt} \qquad \underline{U}_L = jX_L \cdot \underline{I}$

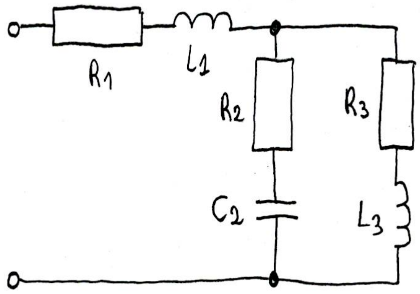


* the voltage across an inductor leads the current by 90° ($\frac{\pi}{2}$)

$\varphi_u = \varphi_i + 90°$

PROBLEM #1

Calculate the equivalent impedance of the circuit shown in the figure.



$R_1 = 10Ω,\ R_2 = 5Ω,\ R_3 = 15Ω,\ L_1 = 50mH,\ L_3 = 200mH,\ C_2 = 1mF,\ \omega = 100\,\frac{rad}{s}$

$X_{L1} = \omega L_1 = 100 \cdot 50 \cdot 10^{-3} = 5Ω$

$X_{L3} = \omega L_3 = 20 \cdot 10^{-3} \cdot 100 = 20Ω$

$X_{C2} = 1/\omega C_2 = \frac{1}{100 \cdot 1 \cdot 10^{-3}} = 10Ω$

$\underline{Z}_1 = R_1 + jX_{L1} = (10 + j5)\,Ω$

$\underline{Z}_2 = R_2 - jX_{C2} = (5 - j10)\,Ω$

$\underline{Z}_3 = R_3 + jX_{L3} = (15 + j20)\,Ω$

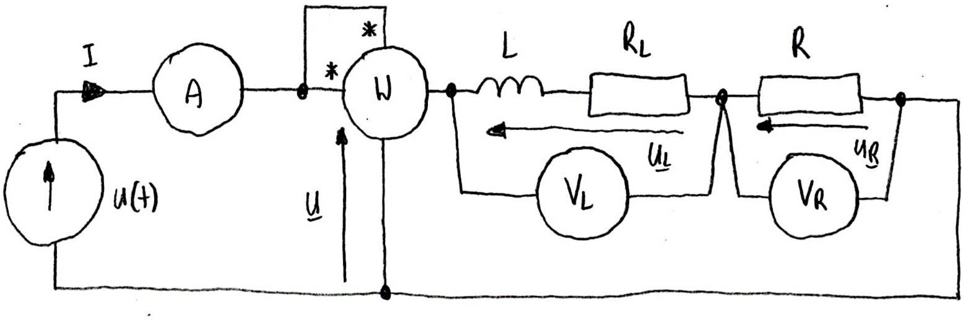
$\underline{Z}_{23} = \frac{\underline{Z}_2 \cdot \underline{Z}_3}{\underline{Z}_2 + \underline{Z}_3} = \frac{(5-j10)(15+j20)}{5-j10+15+j20} = (10 - j\,7.5)\,Ω$

$\underline{Z}eq = \underline{Z}_1 + \underline{Z}_{23} = 10 + j5 + 10 - j7.5 = \boxed{(20 - j\,2.5)\,Ω}$

PROBLEM #2

Calculate meter readings in the circuit shown in the figure.

$u(t) = 230\sqrt{2}\sin \omega t$ V, $L = 0.2$ H, $R_L = 40Ω$, $R = 100Ω$, $f = 50$ Hz



$u(t) = U_{rsm}\sqrt{2}\sin(\omega t + \varphi)$

$\underline{U} = U_{rsm}(\cos\varphi + j\sin\varphi)$

$U_{rsm} = 230 \qquad \varphi = 0°$

$\underline{U} = 230(\cos 0° + j\sin 0°)$

$\underline{U} = 230(1 + 0j) = 230$ V

$X_L = \omega L = 2\pi f L = 2 \cdot 3.14 \cdot 50 \cdot 0.2 = 62.83\,\Omega$

$\underline{Z} = jX_L + R_L + R = 62.83j + 40 + 100 = (140 + 62.83j)\,\Omega$

$\underline{I} = \frac{\underline{U}}{\underline{Z}} = \frac{230}{140 + 62.83j} = (1.37 - 0.61j)\,A \qquad I_A = |\underline{I}| = \sqrt{1.37^2 + (-0.61)^2} = \boxed{1.5\,A}$

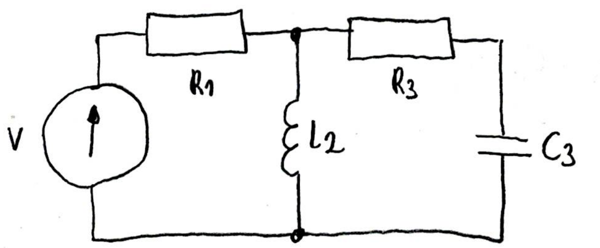
$\underline{U_L} = (R_L + jX_L) \cdot \underline{I} = (40 + 62.83j)(1.37 - 0.61j) = (93.26 + 61.37j)\,V \qquad V_L = |\underline{U_L}| = \boxed{111.64\,V}$

$\underline{U_R} = R \cdot \underline{I} = 100 \cdot (1.37 - 0.61j) = (137 - 61j)\,V \qquad V_R = |\underline{U_R}| = \boxed{149.88\,V}$

$\underline{S} = \underline{U} \cdot \underline{I}^* = 230(1.37 + 0.61j) = (\underbrace{314}_{P} + \underbrace{141j}_{Q})\,VA \qquad P = Re\{\underline{S}\} = \boxed{314\,W}$

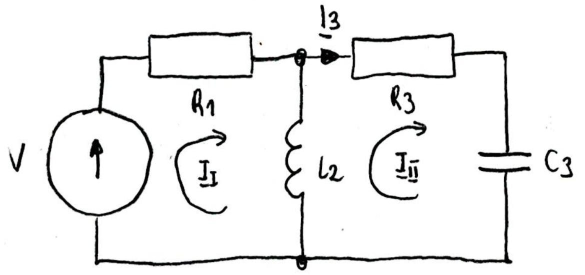
PROBLEM #3

In the circuit shown in the figure, the resistor R_3 has the maximum power $P_3 = 8W$. Check if this is enough for the correct operation of this system.
$V = 24\angle 60°\,V$, $R_1 = 4\Omega$, $X_{L2} = 6\Omega$, $R_3 = 8\Omega$, $X_{C3} = 4\Omega$



$\underline{V} = 24(\cos 60° + j \sin 60°) = (12 + j\,20.7846)\,V$

Method 1 (Loop-Current Method)



$$\begin{cases} \underline{I_I}(R_1 + jX_{L2}) - \underline{I_{II}}(jX_{L2}) = \underline{V} \\ -\underline{I_I}(jX_{L2}) + \underline{I_{II}}(R_3 + jX_{L2} - jX_{C3}) = 0 \end{cases}$$

$$\begin{cases} (4+j6)\,\underline{I_I} - (j6)\,\underline{I_{II}} = 12 + j\,20.7846 \\ -(j6)\,\underline{I_I} + (8+j2) = 0 \end{cases}$$

$$W = \begin{vmatrix} 4+j6 & -j6 \\ -j6 & 8+j2 \end{vmatrix} = (4+j6)(8+j2) - (-j6)(-j6) = 32 + j8 + j48 - 12 + 36 = 56 + j56$$

$$W_{II} = \begin{vmatrix} 4+j6 & 12+j\,20.7846 \\ -j6 & 0 \end{vmatrix} = (4+j6)\cdot 0 - (-j6)\cdot(12 + j\,20.7846) = -124.71 + j\,72$$

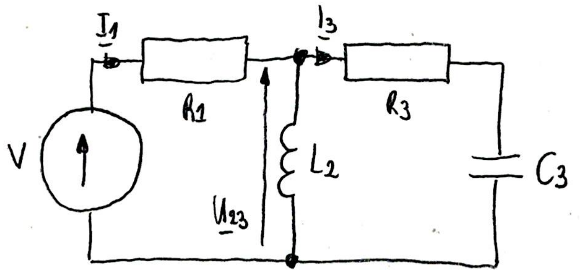
$\underline{I_{II}} = \frac{W_{II}}{W} = \frac{-124.71 + j72}{56 + j56} = (-0.4706 + j\,1.7563)\,A \qquad \underline{I_3} = \underline{I_{II}}$

$\underline{U_3} = R_3 \cdot \underline{I_3} = 8 \cdot (-0.4706 + j\,1.7563) = (-3.7648 + j\,14.0505)\,V$

$\underline{S} = \underline{U_3} \cdot \underline{I_3}^* = (-3.7648 + j\,14.0505) \cdot (-0.4706 - j\,1.7563) = 26.449\,VA \qquad P_{R3} = Re[\underline{S}] = 26.449\,W > 8W$

The resistor has too low maximum power.

Method 2 (Ohm's Law, KCL, KVL)



$\underline{Z}_1 = R_1 = 4\Omega$

$\underline{Z}_2 = jX_{L2} = j6\Omega$

$\underline{Z}_3 = R_3 - jX_{C3} = (8-j4)\Omega$

$\underline{Z}_{23} = \dfrac{\underline{Z}_2 \cdot \underline{Z}_3}{\underline{Z}_2 + \underline{Z}_3} = \dfrac{j6\cdot(8-j4)}{j6+8-j4} = (4.2353 + j\,4.9412)\Omega$

$\underline{Z}_{eq} = \underline{Z}_1 + \underline{Z}_{23} = 4 + 4.2353 + j\,4.9412 = (8.2353 + j\,4.9412)\Omega$

$\underline{I}_1 = \dfrac{\underline{V}}{\underline{Z}_{eq}} = \dfrac{12 + j\,20.7846}{8.2353 + j\,4.9412} = (2.1849 + j\,1.2129)A$

$\underline{U}_{23} = \underline{Z}_{23} \cdot \underline{I}_1 = (4.2353 + j\,4.9412)\cdot(2.1849 + j\,1.2129) = (3.2604 + j\,15.933)V$

$\underline{I}_3 = \dfrac{\underline{U}_{23}}{\underline{Z}_3} = \dfrac{(3.2604 + j\,15.933)}{8-j4} = (-0.4706 + j\,1.7563)A$

$\underline{U}_3 = R_3 \cdot \underline{I}_3 = 8\cdot(-0.4706 + j\,1.7563) = (-3.7648 + j\,14.0505)V$

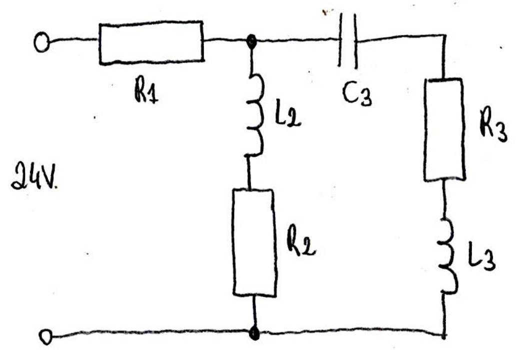
$\underline{S} = \underline{U}_3 \cdot \underline{I}_3^* = (-3.7648 + j\,14.0505)\cdot(-0.4706 - j\,1.7563) = 26.449\ VA$

$P_{R3} = Re[\underline{S}] = 26.449\ W > 8\ W$

The resistor R_3 has too low maximum power.

PROBLEM #4 (this problem was not solved during the class)

The circuit as shown in the figure has been protected by a 6A overcurrent circuit breaker. Check that it will ensure continuous operation of this circuit when supplied with a sine wave voltage of 24 V rms.
$R_1 = 2\Omega$, $X_{L2} = 2\Omega$, $R_2 = 2\Omega$, $X_{C3} = 4\Omega$, $R_3 = 4\Omega$, $X_{L3} = 6\Omega$



$\underline{U} = 24V$

$\underline{Z}_1 = R_1 = 2\Omega$

$\underline{Z}_2 = R_2 + jX_{L2} = (2+2j)\Omega$

$\underline{Z}_3 = R_3 - jX_{C3} + jX_{L3} = 4 - 4j + 6j = (4+2j)\Omega$

$\underline{Z}_{23} = \dfrac{\underline{Z}_2 \cdot \underline{Z}_3}{\underline{Z}_2 + \underline{Z}_3} = \dfrac{(2+2j)(4+2j)}{2+2j+4+2j} = (1.38 + 1.08j)\Omega$

$\underline{Z} = \underline{Z}_1 + \underline{Z}_{23} = 2 + 1.38 + 1.08j = (3.38 + 1.08j)\Omega$

$\underline{I} = \dfrac{\underline{U}}{\underline{Z}} = \dfrac{24}{3.38 + 1.08j} = (6.44 - 2.05)A$ $\quad I = |\underline{I}| = \sqrt{6.44^2 + (-2.05)^2} = \boxed{6.76\,A} > 6A$

The current of an overcurrent circuit breaker is too low.