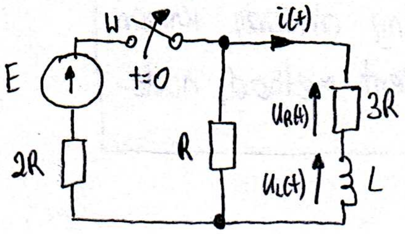
ELECTRICAL CIRCUITS 2 - CLASS NO. 11 (28.05.2024)

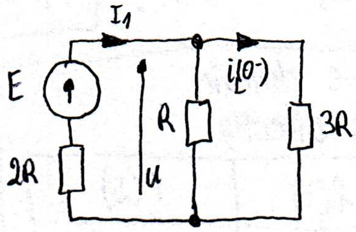
PROBLEM #1

The circuit shown in the figure has been in a steady-state. The switch was open at $t=0$. Find and plot $i(t)$, $u_L(t)$, $u_R(t)$. Use the Laplace transform.



* we start from determining the initial values in the circuit - we solve circuit for initial capacitor voltages ($u_C(0^-)$) and inductor currents ($i_L(0^-)$)
* this require the analysis of a circuit valid for $t<0$ drawn with all capacitors replaced by open circuits and all inductors replaced by short circuits

circuit for $t<0$, $t=0^-$ ($i_L(0^-)=?$)



$$R_{eq} = 2R$$

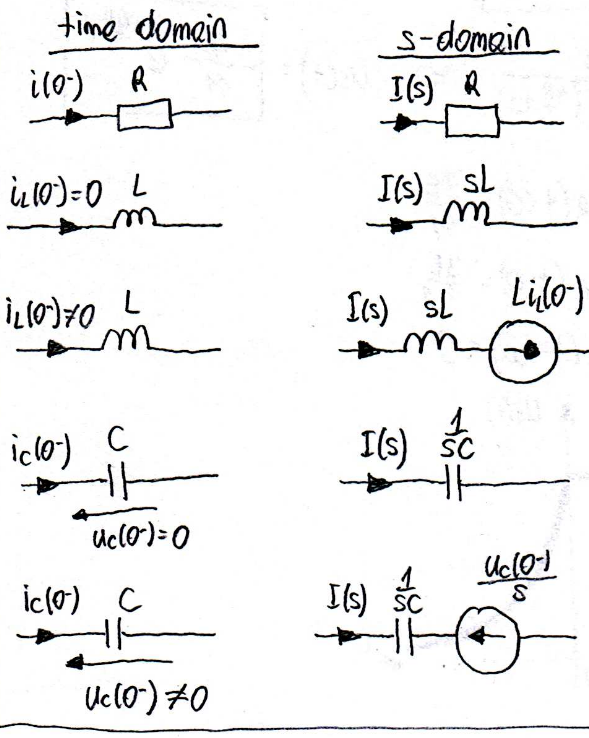
$$I_1 = \frac{E}{R_{eq}}$$

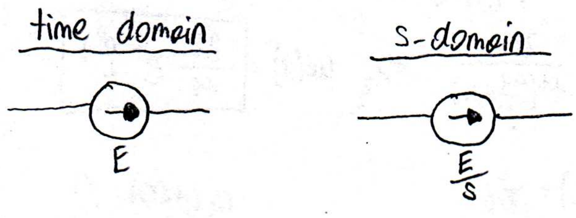
$$R_{eq} = 2R + \frac{R\cdot 3R}{R+3R} = 2R + \frac{3R^2}{4R} = 2R + \frac{3}{4}R = \frac{11}{4}R$$

$$I_1 = \frac{E}{R_{eq}} = \frac{E}{\frac{11}{4}R} = \frac{4E}{11R} \qquad u = I_1 \cdot \frac{R\cdot 3R}{R+3R} = \frac{4E}{11R}\cdot\frac{3R^2}{4R} = \frac{3E}{11}$$

$$i_L(0^-) = \frac{u}{3R} = \frac{3E}{11\cdot 3R} = \frac{E}{11R}$$

* next, we transform the problem from the time domain to the complex frequency domain (that is, s-domain)
* we draw an s-domain circuit by substituting an s-domain representation for all circuit elements

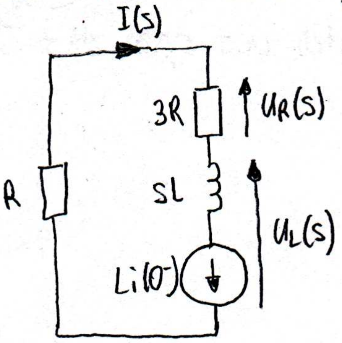




the initial condition of inductor is represented by a voltage source ($Li_L(0^-)$), whose direction is the same as the $i_L(0^-)$ current direction

the initial condition of capacitor is represented by a voltage source $\frac{u_C(0^-)}{s}$, whose direction is the same as the voltage drop $u_C(0^-)$

an s-domain circuit, $t>0$, $i_L(0^-) = i_L(0^+)$



* next, we solve the circuit equations algebraically in the complex frequency domain using already known methods: Ohm's Law, KVL, KCL, loop-current method, node-voltage method, etc.

$$I(s) = \frac{Li(0^-)}{R+3R+sL} = \frac{L\cdot E}{11R(4R+sL)} = \frac{\cancel{L}E}{11R\cancel{L}(4\frac{R}{L}+s)} = \boxed{\frac{E}{11R\left(\frac{4R}{L}+s\right)}}$$

$$U_R(s) = 3R\cdot I(s) = \frac{3\cancel{R}E}{11\cancel{R}\left(\frac{4R}{L}+s\right)} = \boxed{\frac{3E}{11\left(\frac{4R}{L}+s\right)}}$$

$$U_L(s) = sL\cdot I(s) - Li(0^-) = \frac{sLE}{11R\left(\frac{4R}{L}+s\right)} - \frac{LE}{11R} = \frac{LE}{11R}\left[\frac{s}{\frac{4R}{L}+s} - 1\right] = \frac{LE}{11R}\left[\frac{s-\frac{4R}{L}-s}{\frac{4R}{L}+s}\right] = \frac{-4\cancel{R}\cancel{L}E}{11\cancel{R}\cancel{L}\left(\frac{4R}{L}+s\right)} = \boxed{\frac{-4E}{11\left(\frac{4R}{L}+s\right)}}$$

* finally, we transform the solution from the s-domain back to the time domain
* for simple solutions, we can use table of Laplace transform pairs directly

$f(t)$	$F(s)$
$\varepsilon(t)$	$\frac{1}{s}$
$\varepsilon(t-a)$	$\frac{1}{s}e^{-as}$
$\delta(t)$	1

$f(t)$	$F(s)$
$e^{-at}\ (a>0)$	$\frac{1}{s+a}$
$e^{at}\ (a<0)$	$\frac{1}{s-a}$
t	$\frac{1}{s^2}$

$f(t)$	$F(s)$
$\sin\omega t$	$\frac{\omega}{s^2+\omega}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}\ (n=1,2,\dots)$

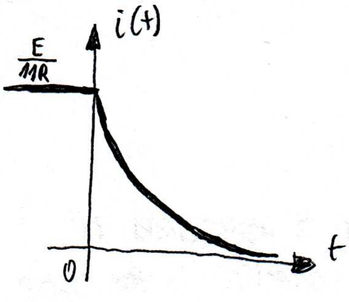
$$I(s) = \frac{E}{11R\left(\frac{4R}{L}+s\right)} \qquad L^{-1}\left[\frac{1}{a+s}\right] = e^{-at} \qquad \Rightarrow \boxed{i(t) = \frac{E}{11R}e^{-\frac{4R}{L}t}}$$

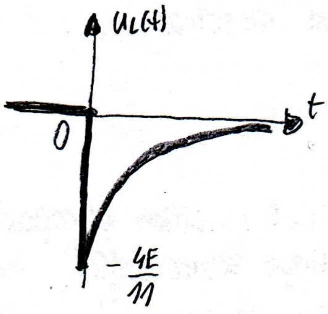
$$U_R(s) = \frac{3E}{11\left(\frac{4R}{L}+s\right)} \Rightarrow U_R(t) = \boxed{\frac{3E}{11}e^{-\frac{4R}{L}t}} \qquad U_L(s) = -\frac{4E}{11\left(\frac{4R}{L}+s\right)} \Rightarrow U_L(t) = \boxed{-\frac{4E}{11}e^{-\frac{4R}{L}t}}$$

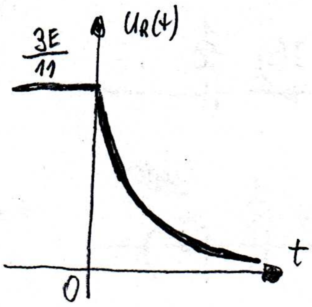
$i(t<0) = \frac{E}{11R}$ $\quad$ $U_L(t<0) = 0$ $\quad$ $U_R(t<0) = \frac{3E}{11}$

$i(t=0) = \frac{E}{11R}$ $\quad$ $U_L(t=0) = -\frac{4E}{11}$ $\quad$ $U_R(t=0) = \frac{3E}{11}$

$i(t\to\infty) = 0$ $\quad$ $U_L(t\to\infty) = 0$ $\quad$ $U_R(t\to\infty) = 0$

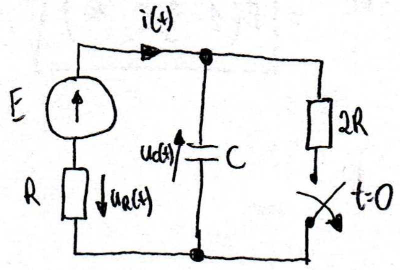




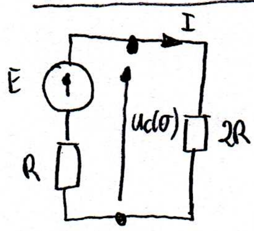


PROBLEM #2

The circuit shown in the figure has been in a steady-state. The switch was open at $t=0$. Find and plot $i(t)$, $u_C(t)$, $u_R(t)$.



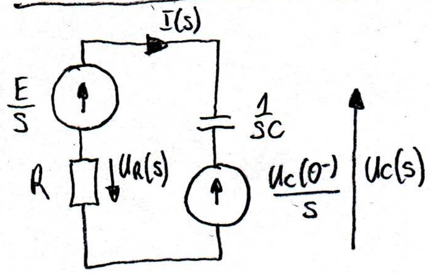
circuit for $t<0$



$$I=\frac{E}{R+2R}=\frac{E}{3R}$$

$$u_C(0^-)=2R\cdot I=2R\cdot\frac{E}{3R}=\boxed{\frac{2E}{3}}$$

circuit for $t>0$ (s-domain circuit)



$$I(s)=\frac{\frac{E}{s}-\frac{u_C(0^-)}{s}}{R+\frac{1}{sC}}=\frac{E-u_C(0^-)}{s\left(R+\frac{1}{sC}\right)}=\frac{E-\frac{2}{3}E}{sR+\frac{1}{C}}=\frac{\frac{1}{3}E}{R\left(s+\frac{1}{RC}\right)}=\boxed{\frac{E}{3R\left(s+\frac{1}{RC}\right)}}$$

$$u_C(s)=I(s)\cdot\frac{1}{sC}+\frac{u_C(0^-)}{s}$$

$$u_C(s)=\frac{E}{3R\left(s+\frac{1}{RC}\right)}\cdot\frac{1}{sC}+\frac{2E}{3s}=\frac{E}{3RsC\left(s+\frac{1}{RC}\right)}+\frac{2E}{3s}=\boxed{\frac{E}{3s(RCs+1)}}+\frac{2E}{3s}=\boxed{\frac{A_1}{s-s_1}+\frac{A_2}{s-s_2}}+\frac{2E}{3s}$$

this transformation requires partial fraction decomposition

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* we have a rational function of s of the form:

$$F(s)=\frac{P(s)}{Q(s)}=\frac{a_m s^m+a_{m-1}s^{m-1}+\ldots+a_1 s+a_0}{b_n s^n+b_{n-1}s^{n-1}+\ldots+b_1 s+b_0}$$

the roots of the polynomial $P(s)$ are called the zeros of the function $F(s)$

the roots of the polynominal $Q(s)$ are called the poles of the function $F(s)$

if $m<n$ and zeros are different from poles:

$$\frac{P(s)}{Q(s)}=\frac{A_1}{s-s_1}+\frac{A_2}{s-s_2}+\ldots+\frac{A_n}{s-s_n} \quad \text{where } A_k=\frac{P(s_k)}{Q'(s_k)},\ k=1,2,\ldots,n$$

$$F(s)=\frac{P(s)}{Q(s)}=\frac{E}{3s(RCs+1)}=\frac{A_1}{s-s_1}+\frac{A_2}{s-s_2}$$

$P(s)=E$

$Q(s)=3s(RCs+1)=3RCs^2+3s \qquad Q'(s)=6RCs+3$

poles $\quad Q(s)=0 \quad 3s(RCs+1)=0$ when $3s=0$ and $RCs+1=0$

$$s_1=0 \qquad s_2=-\frac{1}{RC}$$

$$A_1 = \frac{P(s_1)}{Q'(s_1)} = \frac{E}{6RC\cdot 0 + 3} = \frac{E}{3} \qquad A_2 = \frac{P(s_2)}{Q'(s_2)} = \frac{E}{6\cancel{RC}\cdot\left(-\frac{1}{\cancel{RC}}\right)+3} = \frac{E}{-6+3} = -\frac{E}{3}$$

$$F(s) = \frac{E}{3s} - \frac{E}{3(s+\frac{1}{RC})}$$

$$U_C(s) = \frac{E}{3s} - \frac{E}{3(s+\frac{1}{RC})} + \frac{2E}{3s} = \frac{E}{s} - \frac{E}{3(s+\frac{1}{RC})} \Rightarrow u_C(t) = E - \frac{E}{3}e^{-\frac{1}{RC}t} = \boxed{E\left(1-\frac{1}{3}e^{-\frac{t}{RC}}\right)}$$

$$I(s) = \frac{E}{3R(s+\frac{1}{RC})} \Rightarrow i(t) = \boxed{\frac{E}{3R}e^{-\frac{t}{RC}}}$$

or

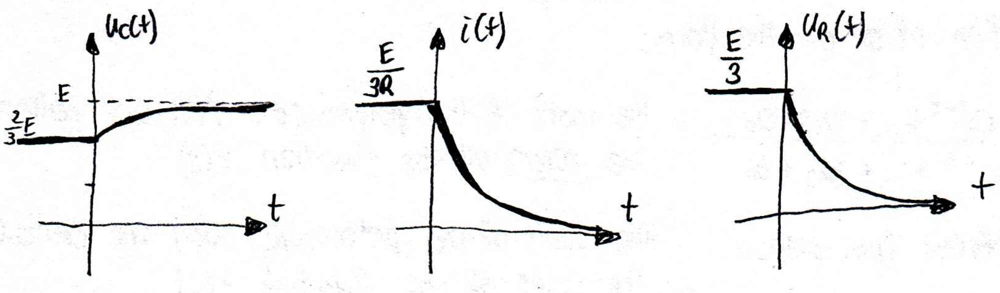
$$i(t) = C\frac{du_C(t)}{dt} = C\frac{d}{dt}\left(E - \frac{E}{3}e^{-\frac{t}{RC}}\right) = \cancel{C}\frac{E}{3}\frac{1}{R\cancel{C}}e^{-\frac{t}{RC}} = \boxed{\frac{E}{3R}e^{-\frac{t}{RC}}}$$

$$U_R(s) = R\cdot I(s) = \frac{E\cancel{R}}{3\cancel{R}(s+\frac{1}{RC})} = \boxed{\frac{E}{3}e^{-\frac{t}{RC}}}$$

or

$$u_R(t) = R\cdot i(t) = \frac{\cancel{R}\cdot E}{3\cancel{R}}e^{-\frac{t}{RC}} = \boxed{\frac{E}{3}e^{-\frac{t}{RC}}}$$

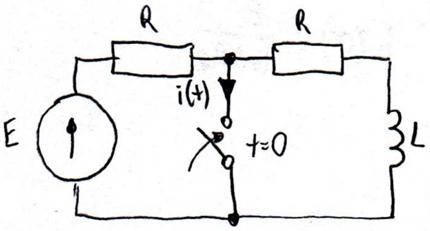
$u_C(t<0) = \frac{2E}{3}$	$i(t<0) = \frac{E}{3R}$	$U_R(t<0) = \frac{E}{3}$
$u_C(t=0) = \frac{2E}{3}$	$i(t=0) = \frac{E}{3R}$	$U_R(t=0) = \frac{E}{3}$
$u_C(t\to\infty) = E$	$i(t\to\infty) = 0$	$U_R(t\to\infty) = 0$



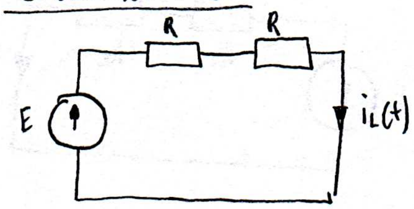
PROBLEM #3

a)

The circuit shown in the figure has been in a steady-state. The switch was close at $t=0$. Calculate and plot $i(t)$. Use the Laplace transform.



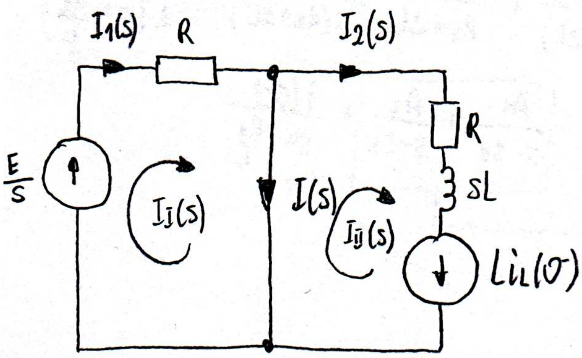
circuit for $t<0$



$$i_L(0^-) = \frac{E}{R+R} = \frac{E}{2R}$$

$$i(t) = 0$$

circuit for $t>0$ (s-domain circuit)



$$\begin{cases} I_I(s)\cdot R = \frac{E}{s} \\ I_{II}(s)(R+sL) = Li_L(0^-) \end{cases} \Rightarrow I_I(s) = \frac{E}{Rs} \quad \Rightarrow I_{II}(s) = \frac{Li_L(0^-)}{R+sL}$$

$$I(s) = I_I(s) - I_{II}(s) = \frac{E}{sR} - \frac{Li_L(0^-)}{R+sL} = \frac{E}{sR} - \frac{L\cdot E}{2R(R+sL)} =$$

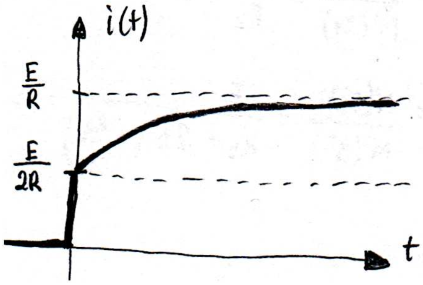
$$= \frac{1}{R}\cdot\frac{E}{s} - \frac{E}{2R(\frac{R}{L}+s)}$$

$$\boxed{i(t) = \frac{E}{R} - \frac{E}{2R}e^{-\frac{R}{L}t}}$$

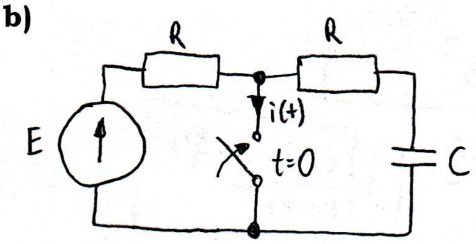
$$\frac{1}{s} \Leftrightarrow \varepsilon(t) \qquad \frac{1}{s+a} \Leftrightarrow e^{-at}$$

$$i(0) = \frac{E}{R} - \frac{E}{2R} = \frac{E}{2R}$$

$$i(t\to\infty) = \frac{E}{R}$$



b)

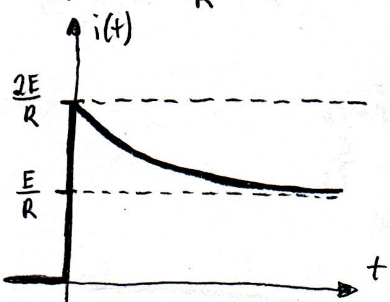


Circuit for $t<0$

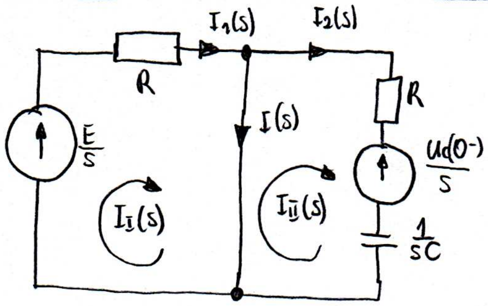
$$u_C(0^-) = E, \quad i(t) = 0$$

$$i(0) = \frac{E}{R} + \frac{E}{R} = \frac{2E}{R}$$

$$i(t\to\infty) = \frac{E}{R}$$



circuit for $t>0$ (s-domain circuit)



$$\begin{cases} I_I(s)\cdot R = \frac{E}{s} \\ I_{II}(s)(R+\frac{1}{sC}) = -\frac{u_C(0^-)}{s} \end{cases}$$

$$\begin{cases} I_I(s) = \frac{E}{sR} \\ I_{II}(s) = -\frac{u_C(0^-)}{s(R+\frac{1}{sC})} = \frac{u_C(0^-)}{sR+\frac{1}{C}} \end{cases}$$

$$\frac{1}{s} \Leftrightarrow \varepsilon(t)$$

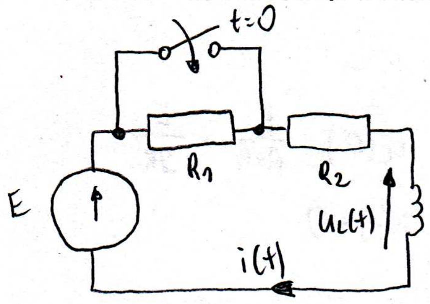
$$I(s) = I_I(s) - I_{II}(s) = \frac{E}{Rs} + \frac{u_C(0^-)}{sR+\frac{1}{C}} = \frac{E}{Rs} + \frac{E}{R(s+\frac{1}{RC})}$$

$$\boxed{i(t) = \frac{E}{R} + \frac{E}{R}e^{-\frac{t}{RC}}}$$

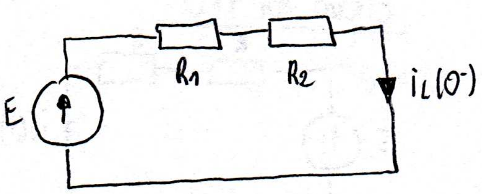
$$\frac{1}{s+a} \Leftrightarrow e^{-at}$$

PROBLEM #4

The circuit shown in the figure has been in a steady-state. The switch was close at $t=0$. Calculate and plot $i(t)$ and $u_L(t)$. Use the Laplace transform.

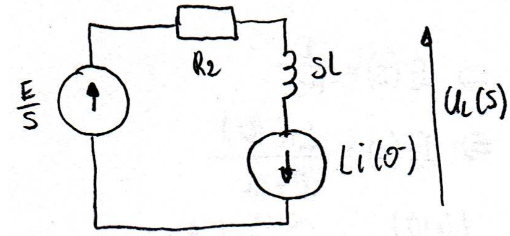


circuit for $t<0$



$$i_L(0^-) = \frac{E}{R_1+R_2}$$

circuit for $t>0$ (s-domain circuit)



$$I(s) = \frac{\frac{E}{s} + Li(0^-)}{R_2+sL} = \frac{E}{s(R_2+sL)} + \frac{Li(0^-)}{R_2+sL} = \frac{E}{s(R_2+sL)} + \frac{\cancel{L}i(0^-)}{\cancel{L}(s+\frac{R_2}{L})} =$$

$$= \boxed{\frac{E}{s(R_2+sL)}} + \frac{i(0^-)}{s+\frac{R_2}{L}} = \boxed{\frac{A_1}{s-s_1} + \frac{A_2}{s-s_2}} + \frac{i(0^-)}{s+\frac{R_2}{L}}$$

$$s(R_2+sL)=0 \quad \Rightarrow \quad s_1=0, \quad s_2=-\frac{R_2}{L}$$

$N(s)=E$

$M(s) = s(R_2+sL) = R_2 s + Ls^2$

$M'(s) = R_2 + 2sL$

$$A_1 = \frac{N(s_1)}{M'(s_1)} = \frac{E}{R_2} \qquad A_2 = \frac{N(s_2)}{M'(s^2)} = \frac{E}{R_2 + 2\cancel{L}\left(-\frac{R_2}{\cancel{L}}\right)} = \frac{E_2}{R_2-2R_2} = -\frac{E_2}{R_2}$$

$$I(s) = \frac{E}{R_2 s} - \frac{E}{R_2(s+\frac{R_2}{L})} + \frac{i(0^-)}{s+\frac{R_2}{L}}$$

$$i(t) = \frac{E}{R_2} - \frac{E}{R_2}e^{-\frac{R_2}{L}t} + i(0^-)e^{-\frac{R_2}{L}t} = \frac{E}{R_2} - \frac{E}{R_2}e^{-\frac{R_2}{L}t} + \frac{E}{R_1+R_2}e^{-\frac{R_2}{L}t} = \frac{E}{R_2} - \left(\frac{E}{R_2} - \frac{E}{R_1+R_2}\right)e^{-\frac{R_2}{L}t} =$$

$$= \frac{E}{R_2} - \left(\frac{ER_1 + \cancel{ER_2} - \cancel{ER_2}}{(R_1+R_2)\cdot R_2}\right)e^{-\frac{R_2}{L}t} = \frac{E}{R_2} - \frac{ER_1}{(R_1+R_2)R_2}e^{-\frac{R_2}{L}t} = \boxed{\frac{E}{R_2}\left(1 - \frac{R_1}{R_1+R_2}e^{-\frac{R_2}{L}t}\right)}$$

$$u_L(t) = L\frac{di(t)}{dt} = L\frac{d}{dt}\left[\frac{E}{R_2} - \frac{ER_1}{R_2(R_1+R_2)}e^{-\frac{R_2}{L}t}\right] = \cancel{L}\left[0 + \frac{ER_1\cancel{R_2}}{\cancel{L}\cancel{R_2}(R_1+R_2)}e^{-\frac{R_2}{L}t}\right] = \boxed{\frac{ER_1}{R_1+R_2}e^{-\frac{R_2}{L}t}}$$

$i(t=0) = \frac{E}{R_1+R_2}$

$i(t\to\infty) = \frac{E}{R_2}$

$u_L(t=0) = \frac{ER_1}{R_1+R_2}$

$u_L(t\to\infty) = 0$

$u_L(t<0) = 0$